Different Concepts of Equations in The Nine Chapters on Mathematical Procedures and in the Commentary on it by Liu Hui (3rd century)

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DIFFERENT CONCEPTS OF EQUATIONS IN *THE NINE CHAPTERS ON MATHEMATICAL PROCEDURES* AND IN THE COMMENTARY ON IT BY LIU HUI (3RD CENTURY)

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The Han text, *The Nine Chapters on Mathematical Procedures*, contains two operations which can be identified as equations[1]: chapter VIII is devoted to the solution of systems of simultaneous linear equations, and problem 20 in chapter IX is solved by means of a quadratic equation[2]. Are these operations, which we regard as being of the same kind conceived of as connected with one another in the Han classic and in its commentary by Liu Hui? This is the question I would like to raise in the present paper. I shall show that, even though these two kinds of equation have features in common, they present essential differences. Moreover, a brief comparison with equations as dealt with in al-Khwarizmi's *Concise Book on the Computation of algebra and al-muqabala* will show affinities between these and equations as treated in the eighth of *The Nine Chapters* rather than with the quadratic equations of chapter IX. In order to prove my point, I shall first examine the content of chapter VIII from this point of view, after which I shall compare the concept of the equation elaborated there with that of the quadratic equation which emerges from chapter IX. Finally, I shall compare both kinds to equations as conceived by al-Khwarizmi.

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1 I have pleasure in expressing my deepest thanks to Barbara Bray without the help of whom this paper could not have been written. Since 1984 I have been working with Guo Shuchun (Academia Sinica, China) on an edition and a French translation of *The Nine Chapters on Mathematical Procedures* (hereafter abbreviated to *The Nine Chapters*) and of the commentaries that Liu Hui (3rd century) and Li Chunfeng (7th century) wrote on it. The outcome of this ten-year collaboration should be published soon. My ideas on this ancient mathematical treatise have undoubtedly been influenced by my collaboration with Guo Shuchun, and I am glad to take this opportunity of acknowledging my debt. Since 1963, when Qian Baocong published his edition of the *Ten Classics in Mathematics*, a Tang compilation which includes *The Nine Chapters*, many editions have been published of this classical text, among them [Guo Shuchun 1990], [Li Jimin 1993]. In this paper, unless otherwise specified, I shall take [Qian 1963] as the basis for my discussion.

2 While *The Nine Chapters* contain only this one quadratic equation ([Qian 1963], pp. 255-6), the commentary ascribed to Liu Hui gives others of the same kind (see problem IX.11, [Qian 1963], pp. 246-8. The edition of this passage is still open to controversy). Moreover, Zhao Shuang, Liu Hui's contemporary, also uses quadratic equations of the same type in his commentary on the Han classic *Zhoubi suanjing* ([Qian 1963], pp. 1-80, and more specifically, pp. 18 *sq*.), and the treatment of algebraic equations developed in the Chinese mathematical tradition within this framework.

I. THE HANDLING OF EQUATIONS IN THE SOLUTION OF SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

In Chapter VIII, named "*fangcheng*",[3] various elements are brought together to form and manipulate an equation. Let us first identify these elements, and then analyse how they are linked to each other, by following step by step the way in which the apparatus to deal with them is presented both in the classic in Liu Hui's commentary.

I.1 THE EQUATION AS A STATEMENT

The first problem, as set out in chapter VIII, reads as follows:

"Suppose that 3 *bing*[4] of high-quality grain, 2 *bing* of medium-quality grain and 1 *bing* of low-quality grain produce (*shi*) 39 *dou*; 2 *bing* of high-quality grain, 3 *bing* of medium-quality grain and 1 *bing* of low-quality grain produce 34 dou; 1 *bing* of high-quality grain, 2 *bing* of medium-quality grain and 3 *bing* of low-quality grain produce 26 dou. Then how much is produced respectively by one *bing* of high-, medium- and low-quality grain?"

Each of the statements setting out the terms of the problem involves the same number of figures and expresses a relationship between them which will prove to be open to transformation. Later on in the chapter, mainly because of the emphasis placed on this aspect by Liu Hui, this gradually emerges as the fundamental form of statement representing an equation, irrespective of variations in the semantics of the relationship and in the function assigned to each term. The semantics of grain production, which provides the framework of 7 out of the 18 problems in the chapter (problems VIII.1 to VIII.6, and then VIII.14), implies an irreversible transformation leading from the grains that are sown to the grain which they yield[5]. Among these problems, some may arise out of a comparison of the products of two such transformations, as in VIII.4:

"Suppose that if we reduce by 1 *dou* 1*sheng* the production of 5 *bing* of high-quality grain, this is *equivalent to* 7 *bing* of low-quality grain, and that if we reduce by 2 *dou* 5

[3] I discuss the meaning of this expression below. As I can only present Chapter VIII very briefly here, I refer the reader to [Chemla 1992] for a bibliography. For the present I leave aside certain aspects treated in my earlier paper, as well as a more detailed presentation of them which will be published elsewhere, in order to concentrate here on the various aspects of the equation itself.

[4] The *bing* is a unit of capacity whose relation to the *dou*, the other unit of capacity used in these statements, is not defined either by *The Nine Chapters*, or by Liu Hui. There may be theoretical reasons for this omission. Other documents give 1 *bing* as equal to 160 *dou*.

[5] The same holds good for the semantics of the terms in problem VIII.12, where horses of various strengths are considered from the point of view of their ability to drag a weight up a slope together. The opposite process, though still irreversible, is involved in problem VIII.16, where we find prefects, ambassadors and members of their escort eating chicken at different unknown rates. Such a situation immediately calls chapter 3 to mind, which cannot be fortuitous.

sheng the production of 7 *bing* of high-quality grain, this is *equivalent to* 5 *bing* of low-quality grain. Then how much is produced by one *bing* of high- and low-quality grain respectively?" (My italics: note the disparity between the data equated.)

However, chapter VIII also contains another kind of problem, the terms of which involve reversible transformations, such as exchanges of goods, as in problem VIII.8:

"Suppose that if a man sells 2 oxen and 5 sheep to buy 13 pigs, he has 1000 coins left, and that if he sells 3 oxen and 3 pigs to buy 9 sheep, he has just enough, and that if he sells 6 sheep and 8 pigs to buy 5 oxen, he has a deficit of 600. Then how much do an ox, a sheep and a pig cost respectively?"[6]

In all our previous examples, transformations are involved, whether irreversible or not (the importance of the difference will emerge below), which lead from one situation to another (from the grain to be sown to its product, from one state of property to another). However the other problems in chapter VIII involve, instead, relationships between unknown measurements or unknown evaluations, as in problem VIII.11:

"Suppose that the price of two horses and an ox exceeds 10000 like the price of half a horse [i.e. $2x + y = 10000 + \frac{1}{2}x$] and that the price of a horse and two oxen does not reach 10000 like the price of half an ox [i.e. $x + 2y = 10000 — \frac{1}{2}y$]. Then how much is the price of an ox and a horse respectively?"[7]

The two statements here deliver *an equality* between the unknowns*,* although in an algorithmic form. In contrast, the corresponding statements constituting the terms of problems such as VIII.1 do not express an equality *stricto sensu*, but rather a relationship between data in terms of production. It is striking in this respect that their semantics is chosen in such a way that the word *shi*, which expresses the amount produced by the grains taken all together, also means "dividend"[8]. And indeed, should there be only one kind of grain, the production would have to be taken as a dividend in order to solve the

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6 The use of terms such as "left", "just enough" and "deficit" reminds us of Chapter VII. In correlation with this, Liu Hui mentions it in his commentary on this problem. Moreover the next problem also reminds us of chapter VII . These are significant parallels, though I cannot comment on them here.

7 Similarly, the evaluation of price provides problems VIII.7, 17 and 18 with their topics (in the case of problem 18, it concerns grains). Problem VIII.10, concerning the circulation of money, is of the same kind. Evaluation of weight is the basis for problems VIII.9 and VIII.15, where again grains are involved. These problems all relate the unknown measurements to definite quantities, be they money or weight. Problem VIII.13, the only indeterminate one in this chapter, refers all unknowns to one of them, as Liu Hui understands it. As a matter of fact, *The Nine Chapters* provide only one solution to the problem, where the value given to the basic unknown is the smallest one to make other unknowns have integral values.

8 It is interesting to recall here Liu Hui's remark concerning *fangcheng*: "This procedure is general, but it is difficult to explain in terms of abstract expressions (*kongyan*). Thus, in order to eliminate this difficulty, it is deliberately linked to grains". The word "expression", which Liu Hui also uses to designate the relationship between the dividend and the other data (see below), might well refer here to the statements attached to equations.

problem. Thus we are led by the term *shi* to consider problem VIII.1 as stating a dividend-divisor relationship, except that it involves more than one divisor[9]. If we read the terms of the problems in chapter VIII with this in mind, we seem to have have here the two extremes between which all statements of relationship between data can be located. Either we find expressions of the kind that "a set of divisors produces a dividend" (VIII.1, VIII.8), or statements of equality (VIII.11), or else intermediate states (VIII.4). In fact, this dividend-divisors link, which lends its form to the expression of an equation, is, as such, liable to transformation: all we need do is to replace one divisor by the dividend that corresponds to it (e.g. replace "3 *bing* of high-quality grain" by "the production of 3 *bing* of high-quality grain"), semantically or numerically, and a new dividends-divisors link occurs, to which division again gives its meaning by extension. Performed on all divisors, this substitution produces equality statements such as the one above. The terms of the problems in *The Nine Chapters* make use of such statements in their various states, and this will be reflected in the various forms under which equations will be worked out as statements. To sum up: the statement corresponding to an equation in Chapter VIII appears to connect data through dividend-divisor relationships, these relationships taking on various semantics according to the problem involved (which may refer to material as well as computational transformations). But each of these statements can be transformed into any of its other states, and is also liable, as a statement, to specific transformations. In his commentary, Liu Hui makes use of this spectrum of possible states of a statement, from the dividend-divisors relationship to the statement of an equality, and of the transformations attached to them taken as statements, to demonstrate the basis on which the mathematical apparatus of Chapter VIII can be developed.

Yet one of these forms of statement, the kind which says that "a set of divisors produces a dividend", as in problem VIII.1, plays a specific part, in that it opens directly on to the other manifestation of the equation, as a configuration of numbers.

I.2 THE EQUATION AS A CONFIGURATION OF NUMBERS

In the first problem in the chapter, our classic text prescribes a set-up of the data on the basis of which the general algorithm for the solution of all such problems is described. The four numbers that appear in the first statement of VIII.1 —3 *bing* of high-quality grain, 2 *bing* of medium-quality grain, 1 *bing* of low-quality grain, 39 *dou*— are arranged in a column where 3 occupies the upper position, 2 the middle, 1 the one underneath, and 39 the lowest. Hence the level in quality of the grain (high, medium and low) appears to be in correlation with a position in an array to be constituted with the data, in the same way as the term *shi* had the double meaning of dividend and

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[9] The relationship is stated —and this is a recurring point— by the mere fact of naming some data "dividend". Again such a link is of an algorithmic type. Moreover, the term *cheng,* which is used to form the title of the chapter, can be understood as a synonym of *fa*, divisor, since they both mean "norm". In order to keep this meaning, I choose to translate it as "measure". All the data entering into the terms of the problems would then be either dividend or divisor. The link of equation to division is in fact fundamental; we shall come back to this point below.

production[10]. Any such statement will likewise, throughout the chapter, be associated with a similar column of numbers, made up of a dividend and as many divisors as the number of unknowns it involves. Liu Hui makes the connection clear in the opening sentences of his commentary : "Things (*wu*) from different groups are brought together though mixed; for each in a line there is a quantity; and their production/dividend (*shi*) is expressed (*yan*) globally".[11]

Actually, in *The Nine Chapters*, the column that represents an equation has no existence in itself: it derives its meaning from the table of numbers in which are arranged the columns linked with the equations constituting the terms of the problem. According to the description given in the classic text, this gives the following table for problem VIII.1:

1	2	3
2	3	2
3	1	1
26	34	39

This kind of table plays a key role in the chapter with which we are concerned, not only because it serves as a basis for the computations of the algorithm, but especially because it provides a framework which, as we shall see, gives to all problems their actual structure[12]. A hint of its importance here can be seen in the fact that the table, or more precisely its form, provides both the chapter itself and its main method with their name: *fangcheng*, which we choose to translate as "measures in a square". This refers to the fact

10 That this double meaning is operative for the commentator Liu Hui can be proved by his using the term *shi* to refer to the constant term, in contexts where no grain production is involved, as in VIII.8, where *shi* is money, and in VIII.15, where it is a weight. Even though *cheng* and *shi* might refer to divisor and dividend, the configuration chosen here for the data is the inverse of what it would be, had the setting of division been used, so far as we can know it. The earliest description of an algorithm for division can be found in the *Sunzi suanjing*, (see [Qian 1963], pp. 282-3), and there is reason to believe this is the same as the one used at the time when *The Nine Chapters* were composed. It sets the dividend above the divisor. In this respect, it might be significant that Qin Jiushao, in his *Shushu jiuzhang* (1247) (see chapter 17, p. 2 for instance), presents a similar version of the algorithm *fangcheng,* where the configuration of numbers is inverted, the *shi* being placed above the other coefficients.

11 Even though each column opposes divisors and a dividend, their set forms a whole, made up of numbers sharing the same nature. As Liu Hui puts it right afterwards : "Let each column be constituted of *lü*", which indicates that they can all be divided or multiplied by the same number without the meaning of the set being changed. On the concept of *lü* and the part it plays in *The Nine Chapters* and Liu Hui's commentary, see [Li 1982] and [Guo 1984].

12 If its columns refer to equations, bringing them together in that way results in having each of its lines convey a mathematical meaning: they appear to gather each the quantities linked with a given unknown. The position in which a number is located in a column hence expresses the unknown to which it is related. Such a meaning given to position reminds one of the representation of a number in a place-value decimal system, such as the one *The Nine Chapters* use. The term that Liu Hui uses for position here (*wei*) is actually the same as the one used to refer to the digits of a number. The upper part of the column could be considered as the development of a number in a system of units that remains unknown. Such a number would constitute the divisor, and the analogy with a division would then be even more precise. Yet for the "quotients" to be determined, the conjunction of a certain amount of divisions is in this case needed. The analogy between number and equation underlies the whole chapter VIII, but I cannot develop that point here.

that the "divisors", *cheng*, when arranged in this way, form a square[13]. This, in a way, describes the setting associated with a given problem, but at the same time it sets the conditions which problems must meet if they are to be solved by this method: they have to be transformed into as many statements of equations as the number of unknowns they involve. The framework of chapter VIII is thus defined by its title.

From the very beginning of the chapter, each linear equation receives two representations, one in the form of a statement among a set of similar statements, the other in the form of a column of numbers embedded in a whole table. Let us see how both are brought into play in the apparatus designed for the solution of systems of simultaneous equations.

I.3 *FANGCHENG:* TRANSFORMATIONS OF EQUATIONS CONSIDERED AS COLUMNS[14]

The general algorithm follows the description of the setting out of the data given in the solution of problem VIII.1. It rests entirely on the table of numbers, which in this case is full, without mention of the statements associated with the equations. Let us translate it before analyzing its main features:

"Multiply the whole middle column by the high-quality grain in the right column, then with it [the right column] eliminate (*chu*)[15] between columns; again multiply the [column] next to it[16] [by the high-quality grain in the right column], and with it [the right column] eliminate between columns too[17]. Afterwards , multiply the whole left column

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13 Liu Hui, in his commentary, makes it clear: "If there are two things, measure twice. If there are three things, measure three times. In every case measure them as many times as there are things." He thus links the length of each line to the length of the upper part of each column, and goes on: " Gather the lines together to make columns: this is why we call this "measures in a square"". This in a way confirms our interpretation of *cheng* as the "divisors", namely as the coefficients arising in the upper part of the table. Such an interpretation matches the opening statement of the chapter *fangcheng* in Yang Hui's *xiangjie jiuzhang suanfa* (1261): "Square is the form of the numbers" (p. 21).

The following sentences from Liu Hui's commentary, though somewhat difficult to interpret, seem to attempt an explanation of how the statements of the equations may be all different and yet all set out together, since they cannot all be real at the same time: "To the left and the right of a column, there is no other that exists at the same time. Moreover, they are expressed (*yan*) when there is ground for it."

14 This algorithm has been discussed in many publications; for a bibliography see [Chemla 1992].

15 This term also designates the central calculation of a division, namely multiplying the quotient by the divisor and subtracting the product from the dividend. On how it helps to define the relationship with the algorithm for division, see [Chemla 1992]. For the present, I merely emphasize the fact that *fangcheng* is linked with division in the terminology used, as well as in the setting of the numbers and in its basic computations.

16 The Chinese text might well be understood here as a plural form (the next ones), which would give a general description for this first step of the algorithm, which aims at eliminating the higher positions in all columns except the right one. Notice how the descriptions "high, middle, low", as well as "right, middle, and left", refer to positions in the table.

17 This "too", like the one below, stresses the iterative character of the procedure, which reproduces the same fundamental step between two columns until it arrives at a triangular matrix. The iterative character occurs at two levels: the same calculations are performed on the various positions in a column —which echoes the place-value setting of the equation—, and the same calculations are performed on the various couples of columns —which echoes the similarity in nature of the vertical arrays.

by the middle-quality grain in the middle column[18], if it is not exhausted, then with it [the middle column], eliminate between columns.

"If the low-quality grain on the left is not exhausted, the higher [term][19] is taken as divisor, the lower as dividend[20]. The dividend (*shi*) gives the dividend of the low-quality grain[21]. If we look for the middle-quality grain, multiply the dividend at the bottom of the middle column by the divisor, and eliminate [from this] the dividend of the low-quality grain[22]. What remains is divided by the number of *bing* of middle-quality grain, which gives the dividend of middle-quality grain[23]. Also, if we look for the high-quality grain, multiply the dividend at the bottom of the right column by the divisor and eliminate [from this] the dividends of the low-and middle- quality grains. What remains is divided by the number of *bing* of high-quality grain, which gives the dividend of the high-quality grain. The dividends are all divided by the divisor, and in each case we arrive at the result in *dou*."

This algorithm thus repeats as many times as necessary the same two fundamental sequences of operations. The first one consists in multiplying a column, say the middle one, by the higher term of another one, say the 3 of the right column, which gives the new following table:

(I)	$3x + 2y + z = 39$	1	6	3
(II)	$6x + 9y + 3z = 102$	2	9	2
(III)	$x + 2y + 3z = 26$	3	3	1
		26	102	39

18 Note that this expression refers to the number now in this position, and not to the number originally placed there. The algorithm uses of this trick (assignation of variables) to bring out the iteration of the same step within the algorithm.

19 Now "higher" and "lower" designate the two not-empty positions left in the left column.

20 By this time, the successive eliminations between columns have reduced one of them to mere division.

21 Its name implies that it requires a divisor by which to be divided.

22 We may be tempted to conclude that the description of the algorithm here lacks generality, and makes use of the fact that, in this case, the coefficient in the last unknown is now 1. But this would not accord with the fact that the same expression occurs below, when the coefficients differ from 1. We may argue rather that naming in this way the number to be subtracted (the dividend/production of the low-quality grain) implies that a multiplication has to be performed which multiplies the amount of low-quality grain in the middle column (a divisor in my interpretation) by the number called earlier "the dividend of the low-quality grain", which transforms this divisor into a dividend/production of the low-quality grain in this column. This fits in with Liu Hui's distinction, in his commentary, between the dividend associated with a position and the dividend associated with one *bing*. Whatever the case, this is the calculation to be performed according to Liu Hui's account. Moreover, the term of dividend refers to a function in the calculation as well: numbers referred to in this way have at one point to be divided by the divisor. This phenomenon — the same name referring in various ways to different numbers within the same algorithm, according to the context and to the point in the computation— often occurs in Chinese mathematical texts.

23 In his commentary, Liu Hui here distinguishes between the dividend associated with a position and that associated with one *bing*.

and then the right equation is subtracted from the middle one as many times as is necessary for the higher term of the middle equation to vanish[24], which gives:

(I) $3x + 2y + z = 39$			1		3
(II) $5y + z = 24$			2	5	2
(III) $x + 2y + 3z = 26$	3	1	1		
			26	24	39

This calculation is thus performed on the table of numbers without reference to the equations as statements. But Liu Hui, examining the correctness of the algorithm, makes explicit the connection between the calculations applied to the columns and the semantics attached to them, and then brings in the things and the dividends that constitute the statements of the equation. This raises two points. First, if the calculation aims at eliminating a position, what about the semantics of the column thereby obtained, the statement attached to it? Secondly, why do such calculations produce columns which are related to the stating of valid relationships? Liu Hui is interested in both, but let us sketch here only his treatment of the first point. The commentator makes it clear that "if, above, a position does not exist, then in this column a thing is also missing. (…) If we cancel the leading position, then below also we delete that which from the dividend/production concerns one thing." (p. 222) Thus, from the point of view of the divisors, such a column, produced by the algorithm, may be interpreted as a statement in which a thing is missing. Moreover, the corresponding impact on the dividend is interpreted as having changed its nature, since, instead of recapitulating the production of three grains, it now concerns only two kinds of grain. Therefore the statement obtained is different in kind from those contained in the terms of problem VIII.1 itself. In other words, a statement which, in another context, may be described as concerning middle- and low-quality grains, is conceived of within this framework as a statement lacking the term concerning high-quality grain. A confrontation with its companion equations turns the absence of a term into a missing term, or else a zero term. Such defective equations, produced at first in column form throughout the algorithm, with their statement then described in the proof, will later in the chapter constitute the terms of new problems. We shall come back to this below. This step will be crucial in order to disentangle the mixed divisors and get a dividend corresponding to a unique divisor. The introduction of the new kind of equation is essential in transforming into divisions the linear equations with which we start.

The second fundamental sequence of calculations, the repetition of which ends in the solution of the problem, enables us, for instance, to deduce y from the following equations:

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24 This is the way in which Liu Hui interprets the "elimination" prescribed by the general algorithm. On this point, see [Chemla 1992]. Note that, if calculation is carried out in this way, the coefficients of the equation must be transformed into integers, which would not be necessary for other descriptions of the flow of operations. Moreover, the numerical values in problem VIII.1 would have allowed a similar but quicker solution: a simple subtraction between (I) and (II) would have eliminated the term in z. This leads us to believe that, although the algorithm is presented within the context of an example, the solution is given in a more general form. This is what Liu Hui asserts: "This is a general procedure".

(I)	$3x + 2y + z = 39$				3
(II)	$5y + z = 24$			5	2
(III)	$36z = 99$		36	1	1
			99	24	39

Instead of eliminating z between equations (II) and (III) —an alternative algorithm which Liu Hui proposes in his comments— *The Nine Chapters* make use of an indirect calculation, understood by Liu Hui as follows: the dividend/production associated with the low-quality grain being now known[25] as 99, we can transform the divisor associated with z in the middle column into its "displayed dividend/production": that is, multiply 99 by 1, and then subtract this displayed dividend from the dividend/production below[26]. Hence, again the calculations proposed in *The Nine Chapters* are explained by making explicit the correlated transformations of the statements in terms of the dividends-divisors relationship attached to the columns. The first of these, the transformation of a divisor into the associated dividend, numerically, performs the kind of transformation typical, as we have seen, of such statements. The second one transforms a dividends-divisor relationship into another dividend-divisor relationship. Each brings into play a kind of transformability of equation taken as a statement[27]. Such transformations as the second turn out to be what comes into focus in problem VIII.2, as we shall see in § I.5. The extension of the framework for *fangcheng*, which immediately appears as a necessity as soon as we admit deficient equations such as those here introduced in the course of Liu Hui's commentary, will be treated in problem VIII.3, to which we now turn.

In conclusion, *The Nine Chapters* attach a representation in the form of a column of numbers to the statement of an equation, after which its calculations bear only on the table of numbers. In contrast with this, Liu Hui keeps going from the column to the statement and back. In this way he describes new kinds of statement of equation and exhibits transformations attached to them —two elements that turn out to be examined immediately afterwards in the next problems in chapter VIII. Is Liu Hui, by doing this,

25 The actual term Liu Hui uses is "having now appeared"; as to the algebra conceived of as the interplay between the known and the hidden, I shall come back to this in another publication.

26 More precisely: the product of 99 by 1 carries the denominator 36, and so, to subtract it from 24 we must reduce the dividend below, 24, to the same denominator. Such is the semantics that Liu Hui ascribes to the multiplication of 24 by 36, as prescribed in *The Nine Chapters*. The subtraction 24.36—1.99 produces, when divided by 5, the dividend/production associated with the middle-quality grain, which still needs to be divided by the denominator 36. Performing the calculation in the same way for the right column will provide three "dividends", each associated with the divisor 36, hence the final division for the three of them. Note that the calculations which have to be performed in Liu Hui's version of the algorithm and in *The Nine Chapters*'s version are the same, but they are grouped differently and hence have different meanings. Moreover, whereas in Liu Hui's version the second part of the algorithm is of the same nature as the first part, in the version proposed by *The Nine Chapters* the two parts differ. Thus Liu Hui carries out in a homogeneous way the elimination of terms from a column, whereas *The Nine Chapters* make use of two different kinds of procedure, one working horizontally, as a computation between columns, and one vertically, as an inner transformation of a column. This leads Liu Hui to comment both on the semantics of column-to-column calculations and on the semantics of transformations within a column.

27 The regrouping, by subtraction, of the two constant terms, after the transformation of the contribution of the low-quality grain, transforms a statement or a column, into equivalent ones. In Liu Hui's version of the algorithm, the same result might be achieved by operations carried out between two columns. Vertical and horizontal calculations are related to one another.

trying to bring out the meaning he sees as being conveyed in the way *The Nine Chapters* organize this sequence of problems?

I.4 Extension/completion of the framework: the general equation[28]

Three statements concerning grain production constitute the terms of problem VIII.3, each of them involving two of the three grains the production of which is to be calculated. Their setting out according to the framework of *fangcheng*, as precribed by *The Nine Chapters*, turns them into deficient equations, missing terms which appear in their companion equations, hence equations such as those analyzed by Liu Hui in the proof of the general algorithm. The corresponding table of numbers, described by Liu Hui, appears as follows:

1		2
	3	1
4	1	
1	1	1

The deficient equations receive from the table of numbers in which they are represented a structure in the form of a column with empty positions. And contrary to the calculations described in *The Nine Chapters* for problem VIII.1, the empty positions will now enter into the column-to-column calculations: void positions become terms in the operations.

Such, then, is the table to which *fangcheng* is to be applied. The upper position of the middle column being already empty, the first elimination bears on the left column, whose upper term has to be cancelled. And elimination between the right and left deficient columns leads immediately to an impossible calculation, since the 1 in the middle position to the right cannot be subtracted from an empty position. Hence the new algorithm, *zhengfushu*, "procedure of the positive and the negative", which *The Nine Chapters* give immediately after and to which we are referred for a description of how to perform *fangcheng* calculations in such cases. It is rather recondite and reads as follows:

" Procedure of the positive and the negative: If the same names are eliminated from one another, different names are increasing each other. When the positive does not enter, make it negative; when the negative does not enter, make it positive.

If different names are eliminated from one another, the same names are increasing each other. When the positive does not enter, make it positive, when the negative does not enter, make it negative."[29]

This passage, we observe, introduces new "names" —"positive" and "negative"— as well as a proposition as to how to compute with them. But *The Nine Chapters* contain nothing more on what they mean, or how they actually occur within a table of numbers, or hence

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[28] Again we shall only deal with this point insofar as the column/statement relationship is put into play in the extension of the framework for *fangcheng*.

[29] See [Chemla 1992] for my interpretation of this passage. As a first approximation, it may be understood as a set of rules facilitating elimination between equations which contain positive as well as negative numbers.

how they can be related to *fangcheng*. These questions are tackled by Liu Hui in the very first sentences of his commentary, and by filling in this gap he builds up a general framework within which to read the whole chapter, and especially a general concept of equation itself. Let us attempt to summarize this part of his commentary. "Now", he says, "two kinds of numbers represented by counting rods are opposed to each other: what one gains and what one loses; we thus must use "positive" and "negative" to name them". This might refer to the first elimination between the left and the right columns: in some cases, the result is something one gains, in other something that one loses, and in case both situations should occur, names need to be introduced to distinguish them. Signs thus make their first appearance in the table during a performance of the algorithm, and do so at the very point where it cannot be applied. From this point on, two problems arise. First, since it now involves positive and negative numbers, the column-to-column elimination in the *fangcheng* procedure can no any longer be performed by mere subtraction at each level, as previously; we have to extend it by making positive and negative eliminate each other, which, for the numbers themselves, Liu Hui concludes, will involve either subtraction or addition. This is where the “procedure of the positive and the negative” comes into play. But Liu Hui temporarily abandons this issue to turn to the second question: positive and negative numbers were introduced during horizontal computations; what about the semantics of the columns which they now form? Liu Hui here states: "If, in the different positions in the same column, we differentiate [numbers] into two categories, for each the difference produced by the subtraction of the sums appears below"[30]. Thus the difference in signs within one column is interpreted as referring to a new kind of relationship between the elements in the column: the dividend below no longer represents the sum of the production attached to each term, but rather the difference between productions attached to the two categories. And so the opposition between positive and negative *signs* within one column is now understood, from a semantical point of view, as opposition between *computations*: terms with the same names are added, terms with different names are subtracted. Such is the kind of statement associated with the new type of column produced by the necessary extension of the algorithm. But now we may follow the same path in the opposite direction, from the statement to the column: then statements presented in this form may be considered as equations and can be represented in the form of a column of positive and negative numbers[31]. Thus Liu Hui has shown how the names of positive and negative occur in all

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30 The dividend below is thus understood as the difference between the dividends (dividends-difference), the first associated with the positive terms and the second with the negative terms. I follow here [Qian 1963]. Yet we could equally well, it seems to me, follow the version in Yang Hui's *Xiangjie jiuzhang suanfa*, and translate as "dividend" (*shi*) instead of "subtract" (See [Guo 1990], note 61, p. 406). The passage would then be translated as "for each one sums up the dividends and their difference appears below.

31 This is how problem VIII.4 (see above) comes to be included within the framework of *fangcheng*. The statements of which it is made up are associated, in a way we shall discuss below, with columns that may be represented in modern symbolism by $5x-7y=11$ and $7x-5y=25$. In the same way, problem VIII.8 (see above) is associated with columns that may be retranscribed in modern terms as $2x+5y-13z=1000$; $3x-9y+3z=0$ and $-5x+6y+8z=-600$. In this case, even the constant term can be negative or zero. This raises an interesting question: the last of the first series of problems concerned with grain production (two equations in two unknowns) should involve, if the columns corresponded exactly with our equations, two negative constant terms. But all sources seem to indicate that *The Nine Chapters* give the constant terms as positive, a point that all standard editions have "corrected" ([Qian 1963], p. 227, [Guo 1990], footnote 98, p. 409, [Li

cases, and so the gap is filled between *fangcheng* and the "procedure of the positive and the negative", concerning how to associate names to numbers. Moreover the area covered by *fangcheng* is extended: the "procedure of the positive and the negative" not only enables us to go on using *fangcheng* to solve such problems as VIII.3, but it can similarly be used as an adjunct to *fangcheng* to start solving such problems as VIII.4. And when Liu Hui goes back to accounting for this extension, it refers to a much wider context than the one he started with.

Let us recapitulate the whole passage. Positive and negative numbers are introduced while using *fangcheng* in some cases; they give rise to new types of column which are interpreted in terms of new kinds of dividend-divisor relationships. Such statements are in turn accepted as constitutive elements of sets that can be solved by the same extension of *fangcheng* as the one needed to complete the computation when positive and negative arise within it. Hence, in his commentary, Liu Hui broadens the scope of *fangcheng* in a way which accounts for the way in which positive and negative are used later on in the chapter by the original text. Missing coefficients were introduced, and then negative coefficients: How many more times is it possible to extend *fangcheng*'s framework? This seems to be the question addressed by Liu Hui when he goes on to claim: "Positive and negative being mixed up with one another, this is *enough* to determine measures from top to bottom; although subtracting and increasing differ, this is *enough* to cause numbers from left to right to communicate; although the "dividends-differences" are of a different sort, this is *enough* to reflect the *lü* with their differences of signs" (my italics). In other words, such an extension makes it possible to apply *fangcheng* in all possible cases and to include any kind of such problems. Following Liu Hui's account of algorithms in the order in which they are given, we have gradually reached a general concept of equation, namely the one that accounts for all the procedures in Chapter VIII of our original text. After this Liu Hui turns to check the correctness of the "procedure of the positive and the negative", and we shall leave him there for the present[32]. Instead, we shall now examine more closely how the various

1993], footnote 38, p. 447). Yet Yang Hui, in his *Xiangjie jiuzhang suanfa*, seems to understand that this is designed to show the limits of the semantics as far as grain production is concerned (the total production must be positive, hence this framework cannot apply to all cases), and that this is connected with the fact that the next problem turns to a semantics in terms of money, within which all cases can be represented. In fact, in the case of a system using two equations, the fact that it retains opposite signs for both constant terms does not mean that the algorithm must produce wrong results, since the final division forgets the signs of its terms. If Yang Hui were right, it would mean that all the sources conform to the original text and that the semantics is not an unimportant matter. Moreover this suggests that neither the composition of the chapter, nor its distribution of the various semantics, is a matter of chance, though I cannot develop this point here. Note that in the same way as equations with missing terms first appeared in the course of the algorithm, and subsequently became statements constituting the terms of new problems, so equations with positive and negative terms here appear within the performance of the algorithm and afterwards are adopted for setting new problems.

32 I will only emphasize here that in checking the correctness of this procedure, Liu Hui makes it clear that the same clauses enable us also to extend the second part of *fangcheng* allowing us to gather together dividends in order to determine unknowns, an extension made necessary in view of the fact that these dividends may have different signs, but an extension that *The Nine Chapters* do not seem to address explicitly. It may also be mentioned that while Liu Hui transformed the algorithm *fangcheng* so that its two phases might be both carried out by the same operation, here Liu Hui similarly shows that the two different clauses given in *The Nine Chapters* may be grouped in a single one.

statements constituting the terms of the problems are reduced to a form linked with the table of numbers on the basis of which the algorithm operates.

I.5 TRANSFORMATIONS OF EQUATIONS AS STATEMENTS

Regarded as a statement, an equation, as we have seen, expresses here a relationship between dividends and divisors. Although such a relationship may take various equivalent forms, one of them appears to play a central role: this is the one which is directly connected with the equation as it appears in the form of a column of numbers, and which we might transcribe in modern terms as $ax + by + \ldots = c$, a, b, c being positive, negative or even equal to zero. All statements of equation contained in the terms of a problem need to be transposed into such a form, which I shall call standard, in order to be transferred to the table of numbers which the algorithm will treat. If we look at Chapter VIII from that point of view, we find that all such transformations of the equation, taken as a statement, are grouped in two classes, one of which is given in *The Nine Chapters* while the other is made explicit by Liu Hui in his commentary.

The first class of transformations is introduced for the solution of problem VIII.2. In modern terms, the statements contained in the terms of the problem are the following:

$$7x - 1 + 2y = 10$$
$$8y + 1 + 2x = 10$$

The two terms expressing addition and subtraction in this case, *yi* and *sun* , are taken up again by the procedure, which states in an abstract way: "That by which we reduce is said to increase; that by which we increase is said to reduce"[33]. Then *The Nine Chapters* explain the meaning of these operations for that precise situation: "If we reduce the dividend by 1 *dou*, the corresponding dividend exceeds 10 *dou*. If we increase the dividend by 1 *dou*, the corresponding dividend does not fill up 10 *dou*."[34] The transformation results in giving both statements their standard form, upon which *fangcheng* can work:

$$7x + 2y = 11$$
$$8y + 2x = 9$$

The same terms recur in the procedures following problems VIII.10 and VIII.11 in the recurring expression: "reduce and increase it", *sunyizhi*. In problem VIII.10, this seems to refer to the transformation of the statement of a relationship whose coefficients involve fractions into a statement without fractions[35]. Problem VIII.11 gathers together

33 In this case, this abstract statement refers to the change of addition in subtraction and *vice versa* when a constant term on one side joins the other in the other side. Liu Hui in his commentary accounts for this way of transforming the statement of the relationship given by the problem into its normal form.

34 Note that this is one of the very rare instances where one finds an explicit explanation of a calculation in *The Nine Chapters*. Moreover such a transformation is exactly what, as Liu Hui showed, accounts for the end of the general algorithm *fangcheng* in the version given by *The Nine Chapters*.

35 Transformation of subtraction into addition and that of division into multiplication are hence grouped in the same class.

both kinds of situation, its terms containing the statement of relationships (see above) which may be transcribed in modern terms as follows:

$$2x + y = 10000 + \frac{1}{2}x$$

$$x + 2y = 10000 — \frac{1}{2}y$$

Liu Hui describes step by step the transformation of these relationships into their standard form. First we obtain

$$[1+\frac{1}{2}]x + y = 10000$$

$$x + [2+\frac{1}{2}]y = 10000$$

and secondly

$$3x + 2y = 20000$$
$$2x + 5y = 20000$$

Here we see that, when it concerns the exchange of addition and subtraction, the transformation "reduce and increase it" applies not only to dividends (i.e. constant terms), but also to divisors (i.e. terms with an unknown). It applies in cases when two terms of the same kind occur on both sides of the relationship, whether they are added or subtracted. At a more general level, it groups together transformations of the statement of a relationship which exchange reduce and increase, whether they take the form of addition/subtraction, or multiplication/division. And it aims at producing what we called a standard form, with integral coefficients.

The second class of transformation, by contrast, bears on relationships where each kind of term occurs once at most, but which are not in standard form; and again it converts them to standard form. This is the case, for example, with the statements involved in the terms of problems VIII.4 or VIII.8, quoted above. In such cases, *The Nine Chapters* merely describe the table of positive and negative numbers corresponding to the relationships stated in the terms of the problem,without making explicit either the statements corresponding to the columns of numbers thus formed or the way they are produced[36]. In all cases, Liu Hui describes the correlative transformation of the statement by an expression such as: "invert these computations", *hu qi suan,* and makes clear how it operates to produce a statement in the standard form[37].

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[36] In modern terms, problem VIII.4 states that $5x — 11 = 7y$, in relation to which *The Nine Chapters* describe the column corresponding to the equation $5x — 7y = 11$. Problem VIII.8 states $2x+5y=13z+1000$, $3x+3z=9y$, $6y+8z=5x-600$, in relation to which *The Nine Chapters* describe the columns corresponding to $2x+5y-13z=1000$, $3x-9y+3z=0$, $-5x+6y+8z=-600$. Only three other problems involve such transformations: VIII.5, VIII.8, VIII.15.

[37] See for instance the first sentence on the commentary to problem VIII.4: "It is said that the dividend/production of 5 *bing* of high-quality grain is too much, and that if we reduce it by 1 *dou* 1 *sheng*, what remains is precisely a quantity equivalent to 7 *bing* of low-quality grain. Therefore we invert these

To summarize our results so far: the algorithm for the solution of systems of simultaneous equations operates on a table of numbers where all the statements of divisors-dividends relationships into which the terms of a problem are transformed —as many as there are unknowns— are represented in the form of a column. This column of numbers is associated, by a one-to-one correspondence, with a statement of the relationship in a standard form (translatable as $ax+by=c$ or else "a set of divisors produce a dividend"). These two representations of the equation are articulated with one another in building the apparatus dealing with equation in chapter VIII, more loosely in *The Nine Chapters* than in Liu Hui's commentary, which relies mainly on the link between both. The necessity, in some cases, of introducing positive and negative numbers into the table of numbers to perform the algorithm results in the introduction of new admissible columns, which in their turn produce new admissible statements of relationship in the terms of problems. These receive their standard form from the column to which they are directly attached: this form is still translatable as $ax+by=c$ with the difference that now addition may become subtraction. Therefore a unique kind of column, involving positive or negative numbers or even zero, may represent any kind of equation. Once the standard forms of the statements of the equations have been transformed into the table of numbers, one just needs to apply the extended unique algorithm to it to get the solution. But how do we arrive at such a standard form? For that purpose, *The Nine Chapters* introduce a class of transformation bearing on the equation taken as the statement of a relationship, and Liu Hui complements this by a second class of the same kind. With these two, any divisors-dividends relationship may be transformed into its standard form, hence it can enter the table in the form of a column of numbers, after which the solution can operate[38]. Such is the apparatus, sketched out by *The Nine Chapters* and supplemented by Liu Hui, which combines the two aspects of equation: the equation as a statement, and transformable as such, and the equation as a column of numbers, similar to a number, which can enter into calculations.

II. RELATIONSHIP TO VARIOUS CONCEPTS OF EQUATIONS WORKED OUT IN ORDER TO DEAL WITH QUADRATIC EQUATIONS

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calculations, subtract one from the other, and take 1 *dou* 1 *sheng* as their difference". Once the statement of the relationship has been transformed in this way it may be identified with the semantics obtained for the new columns produced through the introduction of positive and negative numbers in the "procedure of the positive and the negative". Here Liu Hui quotes that same passage of his commentary on problem VIII.3. Hence, now, instead of going from the sign to the statement, we proceed from the statement to the sign: a statement in this form may be associated with a column of numbers where subtractions are transformed into negative signs, which is precisely the one described in *The Nine Chapters*. Note that, contrary to the previous class of transformation in which the calculations could be performed, this class of transformation merely rearranges the relationship between the components of the statement so that it can take the form which is directly transformable into a column of numbers. To sum up briefly the three different uses of the negative sign we have met so far: it is a mark which allows the extension of the algorithm *fangcheng*, it corresponds to a subtraction in a statement, it takes the meaning of "lack" in contexts with appropriate semantics, such as money. In the table of numbers it is a sign, in the corresponding statement it is an operation. Yet if the algorithm is extended, and if the set of conceivable equations is enlarged by their introduction, no solution of a problem bears a sign.

38 Any calculation based on the column is interpreted by Liu Hui in terms of the statement. Operations on the statement lead to its representation as a column.

II.1 Quadratic equations in China

When we compare the various objects in which we recognize equations in *The Nine Chapters*, we are soon confronted with a striking phenomenon: the way a quadratic equation is introduced and solved in chapter IX is similar to the way in which a general linear equation is produced and dealt with in chapter VIII[39]. In order to argue this point, let us first recall the basic features of the quadratic equation as it appears in the Han classic. As is well known, the fourth of *The Nine Chapters* provides an algorithm for the extraction of the square root of an integer —a procedure based on the decimal place-value representation of the number concerned and carried out on a counting board[40]. This algorithm, which appears, from the way it is described, to be derived from the algorithm of division, calculate the square root digit by digit. After an initial phase in which the first digit a of the square root of A and its order of magnitude 10^n are determined, three numbers are left on the counting board and will serve as the starting point for the computation of the next digits: the top row contains the first digit a, the middle row contains the number remaining once the contribution due the digit a has been removed, namely $A - (a.10^n)^2$, and the lower row contains $2a$, in an appropriate position. The number x, which the algorithm for the extraction of the square-root will compute from this point of the computation onwards, is linked with the numbers left on the counting board by the following relationship:

$$x^2 + 2a \cdot 10^n \cdot x = A - (a.10^n)^2$$

Hence if we delete the first part of the algorithm for the square root extraction and apply the remaining part to an array in the lower row of which is placed α and in the middle row of which is placed β, the algorithm will compute a number x which satisfies the equation:

$$x^2 + \alpha \cdot x = \beta$$

Actually, this is the way in which quadratic equation is introduced in *The Nine Chapters*. It is a numerical operation which involves two terms : the dividend (*shi*) and the so-called "joined divisor" (*congfa*). They are identified with the numbers left on the counting board after the first phase of the square-root extraction, respectively β and α, and the root of the equation is calculated by making use of the algorithm for square-root extraction adequately truncated[41]. Thus the configuration produced during the extraction of a

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39 For a more detailed description of the introduction of quadratic equations in *The Nine Chapters*, see [Chemla 1994].

40 [Chemla 1994a] contains a presentation of this algorithm and a bibliography concerning it.

41 Problem IX.20 is solved by calculating, on the basis of the data, the two terms on the basis of which we are required to "extract the square root". In his commentary, Liu Hui shows that the calculated "dividend" corresponds to the area of a given rectangle whose breadth is equal to the wanted unknown, and that the "joined divisor" corresponds to the difference between its length and its breadth. Hence it is by bringing in the figure of the gnomon that he accounts for *The Nine Chapters'* use of a quadratic equation to solve this

square-root is identified as a mathematical object in which we recognize a quadratic equation. Likewise, the columns involving positive, zero and negative numbers through which the algorithm for the solution of systems of linear equations may have to pass in order to solve a given problem are now themselves introduced as mathematical objects with full rights, which enlarges the class of conceivable linear equations. And in both cases the equations are represented by the configuration which is theirs in the algorithm from which they are extracted, and the algorithms to deal with them are the same as those through which these configurations go when embedded in their original environment.

Since these algorithms both present analogies with division, both kinds of equation are connected, each in its way, with division. First, the configurations of numbers which represent them in a fixed way both oppose a dividend (*shi*) to divisors, all numbers being set in the same column[42]. Secondly, the algorithms that solve equations are in both cases extensions of the algorithm for division. Yet as the nature of these extensions is different, so the equations themselves will differ.

Furthermore, linear and quadratic equations, as they appear in *The Nine Chapters,* present other kinds of opposition. As we saw in the first part of this paper, the column of numbers that represents a linear equation is linked with the statement of a relationship between data in a standard form. Statements of this kind, as such, are open to certain kinds of transformation which bring them to share this standard form, and determine their representation as a configuration of numbers: this is a key point in the apparatus of chapter VIII. On the other hand, no statement of a relationship between data that would be transformable as such goes along with the quadratic equation and consequently no transformation of such statements occurs when we are dealing with these equations; nor is any standard form for such statements to be found. Rather, besides its standard representation as an array of numbers, the quadratic equation seems to be associated by Liu Hui with a standard geometric representation: the figure of the gnomon, which is revealed by the proof he gives of the algorithm for square-root algorithm, and which when identified, in whatever form, leads to the composition of a quadratic equation to

problem. For the relationship between the gnomon and the quadratic equation as conceived of in *The Nine Chapters*, see [Chemla 1994b].

42 In the case of the linear equation, divisors are set above the dividend, in contrast to the probable contemporary setting for division, as we mentioned earlier. Qin Jiushao later inverts the presentation. In the case of square root extraction on the other hand, dividend and divisors are set out in the same way as they were in division. Interestingly enough, for reasons which remain unknown, Li Ye in his *Ceyuan haijing* (1248) inverts this presentation and sets the so-called divisors of the equation above the dividend, as is traditionally the case for linear equations. Later on, in his *Yigu yanduan*, he comes back to the "traditional" setting for algebraic equations. Might his new set-up for equations have come from some analogy between linear and algebraic equations? As we shall see below, there is good reason for asking this question. Whatever the case, the configurations for linear equations and for algebraic equations are similar in *The Nine Chapters*, and the evolution in the lay-out of algebraic equations was to result in their sharing the same configuration: the column of numbers a, b, c, from top to bottom, can then represent, according to the context, either $ax + by = c$ or $cx^2 + bx = a$. Note that in both cases, as in the case of division, the standard configurations distinguish between the constant term and the terms involving unknowns. Again, Qin Jiushao is an exception in this respect, since he chooses to modify the standard representation and associates equations such as the quadratic one above to the column of numbers, from top to bottom : $-a$, b, c.

solve a problem[43]. Such is not the case for the linear equation, which does not seem to be linked with any geometrical form. Therefore, if both kinds of equation share a similar standard representation as an array of numbers, they differ in that one is also represented by a statement in standard form, open to standard transformations, whereas the other is represented by a standard geometrical figure, also open as such to transformations as well.

This crucial opposition was to hold good until the XIIIth century, as far as we know. Then, at a time when algebraic equations were taking on a new form in Chinese mathematical books, other axes of opposition disappeared, as we shall now see. Conceived as it is within *The Nine chapters*, the algebraic equation, in contrast to the linear equation of chapter VIII, lacks generality in two respects. First it fails to embrace in its framework all possible quadratic equations, since only one class of them can be identified with a sub-operation of root-extraction[44]. Secondly, even though an algorithm for cube-root extraction is given in Chapter IV, in parallel with the algorithm for square-root extraction, no cubic equation is associated with it, let alone equations of higher degree. Moreover, again in contrast with the linear equation, the quadratic equation is not given a place-value notation there, since the term in x^2 is missing from the array representing the equation[45].

All these differences were to be annihilated in the XIIIth century. Then, whether in the writings of mathematicians living in southern China, like Qin Jiushao, or in northern China, like Li Ye or Zhu Shijie, algebraic equations of whatever degree were represented, like linear equations, by a pure place-value notation, according to which the column of numbers, from top to bottom[46] $a_1, a_2, \ldots a_n$, represents the equation that we would write as $a_1 = a_2 \cdot x + \ldots + a_n \cdot x^{n-1}$. Hence, whatever their degree, all equations now receive the same notation, and can all be solved by the same algorithm, i.e. the so-called Ruffini-Horner algorithm[47]. Thus, in correlation with the emergence of the place-value notation, algebraic equations seem to have been considered as a unique mathematical object, independent of their degree, and an algorithm appears, which shares the same properties as the algorithm given in *The Nine Chapters,* to solve systems of linear equations. Among these properties, as described above: this algorithm consists of a simple repetition of the same subprocedure along the whole column of numbers representing the equation. Moreover, this subprocedure, applied at first to positive or zero coefficients, was later extended to deal with negative coefficients, henceforward accepted as terms in the equation, in exactly the same way as the algorithm *fangcheng* had been extended in the VIIIth of *The Nine Chapters*. And, thanks to the admission of coefficients

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43 The part played in the evolution of algebraic equations in China by such geometrical representations, and by the transformations associated with them, remains to be described.

44 Namely those which have the form $x^2 + ax = b$, where a and b are both positive numbers.

45 This might be related to the specific form taken by the algorithm for square-root extraction in *The Nine Chapters*. As a result, only equations with 1 as the leading coefficient are considered. Similarly the form taken here by cube-root extraction is such that no cubic equation can be directly extracted from it. Both algorithms were to change in this respect later on. See [Chemla 1994a].

46 Or, from bottom to top, in Li Ye's *Ceyuan haijing*. Moreover Qin Jiushao replaces a_1 by $- a_1$. Needless to say, the symbolical notations are ours: Chinese mathematicians of the XIIIth century wrote down only numerical equations in this way.

47 About the introduction of this algorithm, see [Chemla 1994a]

of any sign, the *same* standard form, opposing a dividend with divisors, was to be extended beyond its previous limits so as to represent any possible equation. In a way this extension reminds one of the linear equations in *The Nine Chapters*: a unique standard configuration of numbers may represent any equation, and the Ruffini-Horner algorithm, when applied to it, has only to be adapted to dealing with the different signs in order to remain valid and unique for all equations. In the same way as, whatever the number of unknowns, there was a single representation available for all linear equations, and a single algorithm to solve their systems as soon as different signs were admitted, there was now a single representation available for all algebraic equations, and a single algorithm to find their "root", assuming that coefficients of any sign are admitted. Hence certain differences which were held to differentiate linear and algebraic equations in *The Nine Chapters* disappeared in the XIIIth century.

II.2 QUADRATIC EQUATIONS IN AL-KHWARIZMI'S BOOK[48]

Even though the *Concise Book on the Computation of algebra and al-muqabala* is devoted to a complete treatment of the quadratic equation, considered as such, the apparatus designed by Al-Khwarizmi reminds one more, in many respects, of the way *The Nine Chapters* and its commentary deal with linear equations than of the way quadratic equations are approached there. To support this statement, let us briefly recall the main characteristics of Al-Khwarizmi's treatment.

The equation, as he conceives it, is the statement of an equality involving three kinds of term which he identifies explicitly: the number, the unknown (sometimes called "thing" *shay*[49] and sometimes "root" *jadr*, and which we call x), and the square of the unknown (named "wealth", *mal*, which refers to x^2). These terms, by composition, give rise to six canonical kinds of equation, whose normal form may be transcribed in modern symbolism as follows:

$$ax^2 = bx, \quad ax^2 = c, \quad bx = c, \quad x^2+bx = c, \quad x^2+c = bx, \quad x^2 = bx+c$$

This means that only strictly positive rational numbers, as we would put it, are admitted as terms entering into the composition of the normal form of an equation. These "normal forms" play a crucial role in al-Khwarizmi's book. On the one hand, each of them is associated with a specific algorithm enabling us to determine the value of the unknown involved in the equation[50]. Such an algorithm is then explicitly proved on the basis of a

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[48] My understanding of the treatment of equations in Al-Khwarizmi's book, which is thought to have been written between 813 and 833, is based on the English translation of the Latin version by Robert of Chester (see [KARPINSKI, Louis Charles 1930]) as well as on [RASHED, Roshdi 1981].

[49] Note the similarity with the term *wu* (thing), found occasionally in *The Nine Chapters,* but frequently used by Liu Hui to refer to an unknown in a linear equation.

[50] These algorithms solve quadratic equations by radicals, a mode of solution that reminds us of what is found on old-Babylonian tablets (among them the famous BM 13901), in Indian books, and alluded to in Diophantus' *Arithmetics* every time a quadratic equation is met with (See [Ver Eecke 1959], problems IV.31, IV.39, V.10, VI.6, VI.10, VI.22). [Rashed 1981] refers to the historical hypotheses to which such similarities have given rise, but stresses how much al-Khwarizmi's algebraic concern differentiates his book

geometrical representation of the terms of the equation[51]. On the other hand, as al-Khwarizmi explicitly states, any statement of an equality involving the basic terms that may enter an equation can be reduced to its corresponding canonical form. This is done by using two kinds of transformation of statements, again explicitly identified: *al-jabr* and *al-muqabala*. These two words, which give the book its title, both refer to various transformations[52]. Expressed in modern symbolism, *al-jabr* designates the operation which leads from such an expression as

$$13x^2—10x = 3$$

to its canonical form

$$13x^2 = 10x+3$$

But it may equally well refer to the transformation which leads from[53]

$$\frac{1}{5}x^2+x = 10$$

to

$$x^2+5x = 50$$

On the other hand, *al-muqabala* transforms such equalities as

$$5x^2+10x = 4x+3$$

into

$$5x^2+6x = 3$$

and more generally may refer, in Al-Khwarizmi's book and those of his followers, to the transformations that, once *al-jabr* has been achieved, brings the equation into its normal form[54], such as the one which transforms the previous one into

$$x^2 + \frac{6}{5}x = \frac{3}{5}$$

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from any of these known previous documents. It is remarkable that such algorithms seem never to have developed in China, where equations have been mainly conceived of within the framework of the algorithm for root-extraction.

51 Al-Khwarizmi's book is the first text that has come down to us where the proof is explicitly given as an item distinct from the statement of the algorithm. These proofs proceed by simple manipulations of areas. Later, most books on algebra written in Arabic were to reproduce the organization of al-Khwarizmi's presentation, though the nature of the proofs evolved noticeably.

52 The set of transformations covered by these two words change according to the various Arabic authors concerned and even within the works of each of them, though the kind of transformation designated is itself unvarying. It is to be noted, however, that they may, for some authors, bear not on the full statement of the equality but on algebraic expressions that are equated. On these points, see [Saliba 1974], which sketches the history of their meaning, and [Rashed 1984], p. 102-4, which comments on these two operations. We shall only allude here to al-Khwarizmi's use of them. [Rashed 1984] shows how these terms were employed to translate the standard operations by which Diophantos was transforming statements, although he stresses that Diophantos' aims differ from al-Khwarizmi's: Diophantos wants to find out an equality between two kinds of terms, whereas al-Khwarizmi wants to exhibit the normal form of an equation.

53 See [Karpinski 1915], p. 107. As [Saliba 1974], p. 194, points out, al-Karaji, in his definition of the operation, describes it as a transformation through which one can get rid of the words "subtraction" and "division", while still preserving the equation itself.

54 On this point, see [Rashed 1984]. See also [Saliba 1974], p. 198.

So goes the apparatus to deal with quadratic equation: thanks to these kinds of transformation, any statement of equality between the three possible kinds of terms is reduced to its "normal form", and hence a specific algorithm can be associated with it and produces its root. In this way, the normal forms embrace all possible quadratic equations, and a complete treatment is thereby given, where equations are conceived of as an object of study in itself. Note how deeply the organization of the presentation here departs from the progressive unfolding of the general framework in the eighth of *The Nine Chapters,* and the commentary upon it by Liu Hui.

Let us now confront the conceptual architecture of these two treatments of equation. We can see that in both cases the key role in the organization of the resolution is played by normal forms of equation. In contrast with the six different types of normal forms that al-Khwarizmi identifies for quadratic equations, *The Nine Chapters* elaborate a unique standard form for linear equations, thanks to the use of addition as well as subtraction, or else of missing terms. Still, in both cases, normal forms can be associated with any conceivable equation.

To the statement of a standard form such as "the set of such divisors produce such a dividend" there corresponds a unique and fixed representation in the form of a column of numbers, composed of positive as well as negative or zero coefficients; this additional level is characteristic of our Chinese source. As was the case in al-Khwarizmi's treatment also, on the one hand the standard forms of all the equations gathered together in one system serve, *via* the table of numbers attached to them, as a basis for a single algorithm that determines all the unknowns. On the other hand, two kinds of transformations can operate on the divisors-dividends-type relationship which constitutes any linear equation, and can transform it into its standard form, on the basis of which its column representation can be produced[55]. Therefore the overall organization of the treatment of equations is similar, the main difference lying in the addition, in the Chinese case, of this supplementary level of a numerical table on the basis of which the algorithm is actually carried out.

We may be tempted to emphasize another difference as more important, namely the difference in the nature of the projects. Indeed, al-Khwarizmi explicitly aims at producing a treatment of general application of quadratic equations, taken as such and with full generality; this is, as [Rashed 1981] stresses, the point at which he departs from his predecessors. In relation to this, al-Khwarizmi considers equations independently of any problematic context within which they may occur, and starts by identifying the primitive terms that may enter their composition. The normal forms that he lists then refer to classes of equations that may be solved in the same way, hence considered from

55 *The Nine chapters* introduce in problem VIII.2 the transformations *sunyi,* which serve to delete dividends or divisors that occur more than once, whether they be added or subtracted, and to transform fractional coefficients into integral ones. When statements involve no term occurring more than once, Liu Hui, on the other hand, names *hu qi suan* the transformation by which they acquire their normal form, of the type "the set of such divisors produce such dividend". This classification of transformations of statements differs from al-Khwarizmi's. Yet in both cases it is interesting to notice that transformations by addition/subtraction may be brought together with transformations by multiplication/division in the same class. Moreover, transformations that delete terms in excess are distinguished from those that rearrange the terms on both sides of the relationship. But since the normal forms that are sought differ, the transformations themselves differ also.

the point of view of their form and not of their value, even though the general algorithms for their solution are themselves described within the framework of particular instances. In addition to this, al-Khwarizmi works out the beginnings of an algebraic calculation designed to enable us to establish an equation in order to solve any given problem, thereby arriving at a comprehensive treatment of quadratic equations and their applications. By contrast, the systems of equations which we have analysed in Chapter VIII of *The Nine Chapters* appear to depend essentially on the framework of the particular problems within which they occur. Hence, if we rely chiefly on such criteria as distinguish al-Khwarizmi's treatment from those of his predecessors, it might seem to indicate that there is no treatment of systems of equations as such in *The Nine Chapters*. But can we deal with this question in that way? It seems to me that such an approach would impose on the Chinese text our expectations of how such a general treatment should be presented, instead of inquiring into the various forms it could have taken given the vagaries of time and space. The whole of Chapter VIII of our Chinese text is devoted to systems of simultaneous linear equations, and it receives its name from the single algorithm proposed to solve them, thus making clear the autonomy of the topic. Moreover, many hints indicate that the algorithms presented for the solution of specific problems in Chapter VIII are in fact meant to have a general value[56]. In addition, we have also noticed more than once in the early part of this paper that the composition of the chapter, i.e. the semantics of the problems, their choice and their arrangement, including the gaps in them, seemed not to be fortuitous, but rather the result of an elaboration conveying various kinds of meaning, some of them made explicit by the commentator. All these remarks indicate that the actual meaning of the whole differs from the sum of the meanings of the pieces taken separately and literally[57]. How was this text meant to be read? How can we determine its actual content? The answer to such questions is beyond the scope of this paper. We would need to compare the composition of the various chapters, and the commentator's reaction to them, if we wished to determine the meaning of each particular problem and of its variations, as well as the meaning of the organization as a whole. Only on such a basis might we be able to answer questions about the precise conception of equations to be found in chapter VIII.

Yet, whatever the conclusions of such a study might turn out to be, an essential difference between al-Khwarizmi's book and *The Nine Chapters* has to be stressed. His way of generating them leads al-Khwarizmi to consider equations as kinds of statement, *without relationship to their solvability*. In relation to this, he distinguishes, when necessary, within a class, between those which are actually soluble, or have more than one "root", and those which are not, by making explicit, on the basis of the normal form, general criteria that define subclasses within the class. By contrast, no mention of unsoluble systems of equations is to be found either in *The Nine Chapters* or in Liu Hui's

56 We have seen, for instance, that the algorithm *fangcheng* might have been performed in a simpler way, had one taken into account the particular numerical values of problem VIII.1, within which it is described. Yet a general though more cumbersome algorithm is preferred.

57 We have already drawn similar conclusions about the significance of the choice of problems that go to make up Chapter VII. See [Chemla 1991].

commentary. This might derive from the fact that in the Chinese work the framework of the equations consists of problems[58].

Despite these differences and the questions that still await answers, the similarities remain striking between the organization of the treatment of linear equations in *The Nine Chapters* and the organization of the treatment of quadratic equations by al-Khwarizmi. On the other hand, the treatment of quadratic equations differs radically between the Chinese and the Arabic sources. This is obvious if we compare *The Nine Chapters* and the *Concise Book on the Computation of algebra and al-muqabala*: the quadratic equation which can be found in chapter IX does not stand for all possible quadratic equations, and is not associated with the statement of an equality; yet its association with the gnomon reminds one of the geometrical apparatus that al-Khwarizmi uses when he proves the algorithms that he sets forth. But this difference in treatment still holds good if we compare later sources, even though, as we have suggested, the treatment of algebraic equations eventually comes closer to the treatment of systems of linear equations. XIIIth century Chinese documents deal with algebraic equations with full generality, though within the framework of problems: algebraic equations of any degree receive the same notation, and are solved by a unique algorithm; but again no statement of equality is associated with them and, as the degrees become higher, they depart from direct geometrical figuration. In contrast with this, algebraic equations in Arabic writings remain the statement of an equality, transformable as such, and keep close to a geometrical interpretation. Various normal forms are distinguished, in correlation with the fact that zero or negative coefficients are not used, and they determine the algorithm to be used to compute the root.

Beyond the differences that oppose them to each other, linear equations, as conceived of within the systems they form, and algebraic equations converge towards each other in XIIIth century China, where they are both represented by the same kind of configurations of numbers, and enter the same kind of algorithms. In this convergence, we may discern the elaboration in China of a concept of equation which takes its fundamental characteristics from the crucial operation of division, of which it constitutes mere variation.

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58 As far as I know, this contrast also holds good for the treatment of algebraic equations, even as late as the XIIIth century, since no discussion is to be found in Chinese sources concerning this possible failure to have roots; whereas Arabic works are always elaborating criteria for solubility. Yet it should be remembered that the fact that a system of linear equations might have more that one solution is shown in problem VIII.13 of *The Nine Chapters*, where Liu Hui qualifies the solutions as *lü*, which is a way of characterizing their nature. Moreover, the fact that an algebraic equation can have more than one "root" (i.e. positive root) will be touched upon in Yang Hui's writings.

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